



FERMILAB-Pub-77/25-THY  
March 1977

## Self-Consistent $\beta$ -functions in Asymptotically Free Massless Field Theory

HENRY D. I. ABARBANEL

Fermi National Accelerator Laboratory, Batavia, Illinois 60510

### ABSTRACT

In massless field theory with asymptotic freedom a spontaneous mass may be generated through renormalization. This mass is non-analytic in the coupling and via the renormalization group is connected with the usual  $\beta$  function. The development of the  $\beta$  function itself in renormalized perturbation theory provides another connection with the spontaneous mass. In a two dimensional field theory of four-fermion coupling which is asymptotically free these connections are exploited to find the infrared behavior of the theory. An argument is made on how the results carry over to all orders of perturbation theory.



# I. INTRODUCTION

In renormalizable field theories with no mass scales in the Lagrangian, a scale is well known to be present, nevertheless, because of the necessity to specify one during the renormalization procedure. The mass scale  $m$  which enters the renormalized theory is constrained by the renormalization group to take the form<sup>1</sup>

$$m = \mu \exp - \int^g dx/\beta(x) \quad , \quad (1)$$

where  $\mu$  is the common point in momentum space where the renormalized Green's functions are defined.  $g$  is the renormalized coupling constant which is defined at  $\mu$ , and  $\beta(g)$  is the usual renormalization group function which determines how  $g$  depends on  $\mu$ .

Theories with asymptotic freedom have the feature that the mass scale in (1) has an essential singularity at  $g = 0$  and will be missed at every order of perturbation theory. If, as in non-abelian gauge theories with few enough fermions,  $\beta(g) = -b_0 g^3 + O(g^5)$  for small  $g$ , then near  $g = 0$

$$m = \mu \exp - 1/2b_0 g^2 \quad (2)$$

To the extent that one is interested in the small  $g$  behavior of such asymptotically free theories the presence of an exponentially small mass term is likely to be of little, if any, interest. After all, the small  $g$  regime of asymptotically free theories is the regime of ultraviolet behavior where presumably all masses, not only exponentially small ones, are unimportant.

If one is interested in exploring the infrared behavior of these theories, this scenario is likely to be quite the reverse. In perturbation theory to every order, no mass is present, and as momentum integrations explore zero momentum no mass will tame the various divergences. However, the presence of a mass scale in the renormalized theory is sure to essentially alter the actual infrared behavior. In particular the function  $\beta(g)$  is likely to be significantly changed in its behavior for large  $g$  which in an asymptotically free theory governs infrared questions.

In this brief note we explore the effect of non-analytic terms such as (2) on the determination  $\beta(g)$ . Our calculations are carried out in a massless fermion theory with four fermion interactions at 4 space and 1 time dimension. The idea is that the mass (1) develops since a scalar field "representing" a fermion-anti-fermion operator develops an expectation value in the vacuum. This mass enters the graphical determination of  $\beta(g)$  as a power series in  $g$  with  $m$  fixed. With this connection between  $\beta(g)$  and  $m$  we are able to use (1) to eliminate  $m$  and solve for  $\beta(g)$ . We call this the self-consistent  $\beta$  function. Indeed, as expected, the small  $g$  behavior of the theory is unaltered. The finite  $g$  behavior is markedly different from what perturbation theory would suggest.  $\beta(g)$  develops an infrared stable zero where the anomalous dimensions vanish. Leaning on the fact that in the neighborhood of this infrared stable zero the mass (1) is infinite, we argue that this feature of  $\beta(g)$  is true in all orders of perturbation theory.

We turn now to the specific theory we will use. Comments on the lessons which we might learn from this example are left to the last paragraph.

## II. THE FOUR FERMION THEORY IN TWO DIMENSIONS

We begin with the four fermion Lagrangian<sup>2</sup> with N component fermions

$$\mathcal{L}_0 = \bar{\psi}_0 i \not{\partial} \psi_0 + \frac{1}{2} g_0^2 (\bar{\psi}_0 \psi_0)^2 \quad (3)$$

which gives rise to the generating functional of fermion Green's functions  $W[\eta, \bar{\eta}]$  via

$$e^{iW[\eta, \bar{\eta}]} = \int d\psi_0 d\bar{\psi}_0 e^{i \int d^D x [\mathcal{L}_0 + \bar{\eta} \psi_0 + \bar{\psi}_0 \eta]}, \quad (4)$$

where we indicate here that we work in D dimensions to regulate divergences.

The four fermion theory can be made a bit more familiar looking by introducing the scalar field  $\sigma_0(x)$  through<sup>2</sup>

$$e^{iW[\eta, \bar{\eta}]} = \int d\bar{\psi}_0 d\psi_0 d\sigma_0 e^{i \int d^D x \left\{ -\sigma_0^2/2 - g_0 \sigma_0 \bar{\psi}_0 \psi_0 + i \bar{\psi}_0 \not{\partial} \psi_0 + \bar{\eta} \psi_0 + \bar{\psi}_0 \eta \right\}}. \quad (5)$$

Now we are dealing with the coupled  $\sigma_0, \psi_0$  Lagrangian

$$\mathcal{L}_0 = \bar{\psi}_0 i \not{\partial} \psi_0 - \sigma_0^2/2 - g_0 \sigma_0 \bar{\psi}_0 \psi_0 \quad (6)$$

We anticipate that under certain circumstances the field  $\sigma_0$  will develop a vacuum expectation value so we write

$$\sigma_0(x) = v_0 + \chi_0(x) \quad (7)$$

and have the shifted Lagrange function

$$\mathcal{L}_0 = \bar{\Psi}_0 (i\not{\partial} - g_0 v_0) \bar{\Psi}_0 - g_0 \chi_0 \bar{\Psi}_0 \Psi_0 - \chi_0^2/2 - v_0 \chi_0 \quad (8)$$

Our immediate goal is to find under what circumstances  $v_0 \neq 0$ . To this end we introduce a source term  $J_0 \chi_0$  in (8) and integrate over the fermion fields to define  $W[\eta, \bar{\eta}, J_0]$  as

$$\begin{aligned} \exp i W[\eta, \bar{\eta}, J_0] = & \int d\chi_0(x) \exp i \int d^D x \left\{ -\frac{\chi_0^2(x)}{2} - v_0 \chi_0(x) - i \operatorname{tr} \log(i\not{\partial} - g_0 v_0 - g_0 \chi_0(x)) \right. \\ & \left. + \bar{\eta} (i\not{\partial} - g_0 (v_0 + \chi_0(x)))^{-1} \eta + J_0 \chi_0 \right\} \quad (9) \end{aligned}$$

leading us to consider the effective Lagrangian for the scalar field in the absence of fermion sources

$$\mathcal{L}_0(\chi_0) = -\frac{\chi_0^2}{2} - v_0 \chi_0 - i \operatorname{tr} \log \left( 1 - \frac{1}{i\not{\partial} - g_0 v_0} g_0 \chi_0(x) \right) + J_0 \chi_0 \quad (10)$$

Our fermion field has  $N$  components. In the limit  $N$  large,  $g_0^2 N$  fixed, the integration over fermion fields yields a very good approximation to the whole functional  $W[\eta, \bar{\eta}, J_0]$ ,<sup>2</sup> when  $\chi_0$  is taken to be the solution

to the classical equations of motion implied by (10). For our present purposes the large  $N$  limit is not needed.

Now we would proceed in the evaluation of the generating functional  $W[\eta, \bar{\eta}, J_0]$  by expanding the path integral (9) about the extremum of the Lagrangian (10) treated as classical. Unfortunately we are prevented from doing that by the divergences (at  $D = 2$ ) of the  $\text{tr log}$  term in  $\mathcal{L}_0(\chi_0)$ . So first we must renormalize it to account for the fermion loops we have evaluated.

The theory involving scalars,  $\chi_0$ , and fermions,  $\psi_0$ , defined by (8) is renormalizable in  $D = 2$  dimensions via the rescalings

$$\begin{aligned} \chi &= \chi_0 Z_3^{-\frac{1}{2}}, \quad v = v_0 Z_3^{-\frac{1}{2}}, \quad J = J_0 Z_3^{\frac{1}{2}}, \\ \psi &= \psi_0 Z_1^{\frac{1}{2}}, \quad g = g_0 Z_1 Z_3^{\frac{1}{2}} Z_g^{-1}. \end{aligned} \quad (11)$$

These renormalization constants are determined as follows: the bare  $\chi$  propagator is

$$D_0(p^2) = -i; \quad (12)$$

require the renormalized propagator to satisfy

$$D_R(p^2) \Big|_{p^2 = -\mu^2} = -i; \quad (13)$$

this determines  $Z_3$ .

The bare fermion propagator is

$$iS_0^{-1}(p) = \not{p} - g_0 \gamma_0 \quad ; \quad (14)$$

require of the renormalized propagator

$$\left. \frac{\partial}{\partial p} \text{tr } i \not{p} S_R^{-1}(p) \right|_{p^2 = -\mu^2} = 1 \quad ; \quad (15)$$

this determines  $Z_1$ .

The bare two-fermion-scalar vertex function is

$$\Gamma_0 = -i g_0 / (2\pi)^{D/2} \quad ; \quad (16)$$

require of the renormalized vertex function

$$\left. \text{tr } \Gamma(p_i) \right|_{p_i^2 = -\mu^2} = -i g \text{tr } \mathbb{1} / (2\pi)^{D/2} \quad ; \quad (17)$$

this gives  $Z_g$ .  $\text{tr } \mathbb{1}$  is the trace of the unit matrix in D dimensions.

At the level of including only one fermion loop which is what the integration over fermion degrees of freedom means, we have

$$Z_1 = Z_g = 1 \quad , \quad (18)$$

and

$$Z_3 = 1 + N g_0^2 K_D (D-1) \Gamma(1 - \frac{D}{2}) \int_0^1 dy [g_0^2 v_0^2 + y(1-y)\mu^2]^{D/2-1} \quad (19)$$

in D dimensions with

$$K_D = \text{tr} \frac{1}{2} \pi^{D/2} / (2\pi)^D \quad . \quad (20)$$

This expression for  $Z_3$  arises from the one loop correction to the  $\chi$  propagator shown in Figure 1.

Now we are prepared to return to the effective Lagrangian (10) which has fermion degrees of freedom integrated out. As noted we wish to treat this as a classical Lagrange function. So we seek a constant (slowly varying really)  $\chi$  such that

$$\left. \frac{\partial \mathcal{L}}{\partial \chi} \right|_{\chi=0} = 0 \quad (21)$$

which says that  $\langle \chi \rangle = 0$ , and

$$\left. \frac{\partial^2 \mathcal{L}}{\partial \chi^2} \right|_{\chi=0} \leq 0 \quad (22)$$

which says the effective potential,  $-\mathcal{L}$ , is a minimum. For constant  $\chi$ , the renormalized  $\mathcal{L}$  is

$$\begin{aligned}
 \mathcal{L}(\chi) = & -\frac{\chi^2}{2} \left\{ 1 + N g^2 K_D (D-1) \Gamma(1 - \frac{D}{2}) \left[ \int_0^1 dy \left[ g^2 v^2 + y(1-y)\mu^2 \right]^{D/2-1} \right. \right. \\
 & \left. \left. - (gv)^{D-2} \right] \right\} \\
 & - \chi \left\{ v + N g^2 v K_D \Gamma(1 - \frac{D}{2}) \left[ (D-1) \int_0^1 dy \left[ g^2 v^2 + y(1-y)\mu^2 \right]^{D/2-1} - (gv)^{D-2} \right] \right\} \\
 & + N K_D (gv)^D \frac{\Gamma(1 - D/2)}{D} \left\{ \left(1 + \frac{\chi}{v}\right)^D - \left(1 + \frac{D\chi}{v} + \frac{D(D-1)}{2} \frac{\chi^2}{v^2}\right) \right\} + J\chi \quad (23)
 \end{aligned}$$

The condition  $\langle \chi \rangle = 0$  means

$$v \left\{ 1 + \frac{2\lambda^2 K_D \Gamma(2 - D/2)}{2 - D} \left[ (D-1) \int_0^1 dy \left[ F(\lambda)^2 + y(1-y) \right]^{D/2-1} - F(\lambda)^{D-2} \right] \right\} = J \quad (24)$$

where the dimensionless coupling  $\lambda = N^{\frac{1}{2}} g(\mu)^{D/2-1}$  has been introduced and

$$F(\lambda) = \frac{gv}{\mu} \quad (25)$$

The stability condition (22) requires the following, when the source  $J = 0$ :

If  $v = 0$ , when  $J = 0$ , that is, no vacuum expectation value for the original scalar field

$$\lambda^2 \leq \frac{(D-2)\Gamma(D)}{2(D-1)K_D \Gamma(2 - D/2)\Gamma(D/2)^2} = \lambda^2_{\text{CRITICAL}} \quad (26)$$

If  $\lambda^2 > \lambda_{\text{CRITICAL}}^2$ ,  $v \neq 0$ , when  $J = 0$ . For  $D = 2 + \epsilon$ ,  $\lambda_{\text{CRITICAL}}^2 = \pi\epsilon$ .

This is just a reminder of the known fact that in  $D = 2 + \epsilon$ , there is a phase transition at a critical value of  $\lambda^2$  which is order  $\epsilon$ . Since we are interested in  $D = 2$ ,  $\lambda_{\text{CRITICAL}}^2 = 0$ ; so we always have  $v \neq 0$ .

Now we come to the connection of these formulae, which are essentially the determination of the fermion mass  $m_F = gv = \mu F(\lambda)$ , with the renormalization group function  $\beta(\lambda)$ . Note that

$$gv = g_0 v_0 Z_1 Z_g^{-1}, \quad (27)$$

so

$$\mu^2 \frac{\partial}{\partial \mu} \left[ g v Z_1^{-1} Z_g \right] \Big|_{g_0, v_0 \text{ fixed}} = 0 \quad (28)$$

which translates to

$$\left[ \frac{1}{2} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma_1 + \gamma_g \right] F(\lambda) = 0, \quad (29)$$

where

$$\gamma_1 = \mu^2 \frac{\partial}{\partial \mu} \log Z_1 \Big|_{g_0, v_0 \text{ fixed}}, \quad (30)$$

$$\gamma_g = \mu^2 \frac{\partial}{\partial \mu} \log Z_g \Big|_{g_0, v_0 \text{ fixed}}, \quad (31)$$

and

$$\beta(\lambda) = \mu^2 \frac{\partial}{\partial \mu} \lambda \Big|_{g_0, v_0 \text{ fixed}} \quad , \quad (32)$$

so

$$\log F(\lambda) = \int^\lambda \frac{d\lambda'}{\beta(\lambda')} \left[ \gamma_1 - \gamma_g - \frac{1}{2} \right] \quad . \quad (33)$$

At the present level of approximation  $\gamma_1 = \gamma_g = 0$ , so we have the connection between  $F(\lambda)$  and  $\beta(\lambda)$  we seek

$$-2 \frac{d \log F(\lambda)}{d\lambda} = \beta(\lambda)^{-1} \quad . \quad (34)$$

This is the realization in this problem of the connection of the "spontaneous" fermion mass  $\mu F(\lambda)$  with  $\beta(\lambda)$ . As suggested in the introduction the connection of  $F(\lambda)$  and  $\beta(\lambda)$  and the determination of  $F(\lambda)$  by the stability arguments (22) and the vacuum expectation value condition  $\langle \chi \rangle = 0$  (or  $\langle \sigma \rangle = v \neq 0$ ) is the key relation in this paper.

A slightly different view of the  $F(\lambda)$ ,  $\beta(\lambda)$  relation is gotten by directly evaluating  $\beta(\lambda)$  from (32) which to the present accuracy is

$$\beta(\lambda) = \frac{(D-2)}{4} \lambda - \frac{\lambda^3 (D-1)}{2} K_D \Gamma(2 - \frac{D}{2}) \int_0^1 \frac{dy y (1-y)}{[F^2(\lambda) + y(1-y)]^{2-D/2}} \quad . \quad (35)$$

This is the expansion of  $\beta(\lambda)$  in  $\lambda$  for fixed  $F(\lambda)$  (or fermion mass if you like). All the subtle non-analytic behavior in  $\lambda$  is in the function  $F(\lambda)$ .

The connection (34) and (35) between  $F(\lambda)$  and  $\beta(\lambda)$  may be integrated to give at  $D = 2$

$$\sqrt{1 + 4F(\lambda)^2} \log \left[ \frac{\sqrt{1 + 4F(\lambda)^2} - 1}{\sqrt{1 + 4F(\lambda)^2} + 1} \right] = -\frac{2\pi}{\lambda^2}, \quad (36)$$

which, to repeat once more, is the condition from (24) that  $v \neq 0$ , when  $J = 0$ .

From this equation for  $F(\lambda)$  we learn that for small  $\lambda$

$$F(\lambda) \underset{\lambda \rightarrow 0}{\sim} \exp - \pi/\lambda^2, \quad (37)$$

and, of course,

$$\beta(\lambda) \underset{\lambda \rightarrow 0}{=} -\frac{\lambda^3}{4\pi} + \lambda e^{-2\pi/\lambda^2} + \dots, \quad (38)$$

showing the explicit non-analytic behavior in  $\lambda$ . As  $\lambda$  increases, however, one finds that near  $\lambda^2 = \pi$

$$F(\lambda) \underset{\lambda^2 \rightarrow \pi}{\sim} \frac{\lambda}{(12(\pi - \lambda^2))^{\frac{1}{2}}} \quad (39)$$

and  $F(\lambda)^2$  is complex for  $\lambda^2 > \pi$  indicating another phase (Figure 2).

Near  $\lambda^2 = \pi$ ,  $\beta(\lambda)$  behaves as

$$\beta(\lambda) = -\frac{(\pi - \lambda^2)}{2\lambda}, \quad (40)$$

indicating it has an infrared stable zero (Figure 3). In the neighborhood of  $\lambda^2 = \pi$  we find for the anomalous dimension of the  $\chi$  field

$$\gamma_3(\lambda^2) = \mu^2 \frac{\partial}{\partial \mu} \log Z_3 \Big|_{g_0, v_0 \text{ fixed}} \quad (41)$$

$$\lambda^2 \approx \pi - \frac{1}{2\pi} (\pi - \lambda^2) \rightarrow 0 \quad (42)$$

So at  $\lambda^2 = \pi$  the fermion mass has become infinite; the theory is infrared stable; and canonical in the sense that all anomalous dimensions vanish.

### III. DISCUSSION

The most immediate question one may ask about the present calculation is to what extent it is limited to the approximation of one fermion loop or equivalently to lowest order perturbation theory in  $\lambda$  for  $\beta(\lambda)$  with  $F(\lambda)$  fixed. Since  $F(\lambda)$  becomes infinite at  $\lambda^2 = \pi$ , higher numbers of fermion loops are more and more strongly suppressed for this coupling. At any order of perturbation theory in  $\lambda$  with  $F(\lambda)$  fixed, then,  $\beta(\lambda)$  will vanish at  $\lambda^2 = \pi$ . Of course, the behavior at  $\lambda \approx 0$  will also remain unaltered. Between  $\lambda = 0$  and  $\lambda^2 = \pi$  the detailed shape of  $\beta(\lambda)$  will be changed.

Perhaps it is worthwhile to close with a repetition of the idea of the self-consistent  $\beta$  function. In a massless theory with asymptotic freedom there may well be mass generation through renormalization. This mass must take the form

$$m(g) = \mu \exp - \int^g dx / \beta(x)$$

as dictated by the renormalization group. Now  $\beta(g)$  itself is determined by renormalized perturbation theory which is an expansion in  $g$  with fixed  $m(g)$ . This connection of  $m(g)$  and  $\beta(g)$  determines a self-consistent  $\beta$  function. For small  $g$ ,  $\beta(g)$  must be exactly the familiar perturbative result and  $m(g) \rightarrow 0$  rapidly. This is because small  $g$  governs the ultraviolet behavior and masses are not important when they are smooth enough. For  $g$  away from zero, order unity or even large, one is probing the infrared region and masses are certainly important. We have seen in this note an explicit example of this phenomenon.

Clearly the most interesting asymptotically free theory is massless QCD. One may expect to be able to formulate a self-consistent  $\beta$  function there as well. Unfortunately the operator whose expectation value signals mass generation must be composite and thus in a technical sense the problem is certainly more difficult than the example treated here. One may well imagine, however, that the explicit presence of  $\exp - 1/g^2$  terms in non-abelian gauge theories studied in expansion around instantons<sup>3</sup> is a signal of just the kind of mass generation one needs to formulate a self-consistent  $\beta$  function scheme.

#### ACKNOWLEDGMENTS

This work was done when I was a visitor at various Japanese universities under the auspices of the Japan Society for the Promotion of Science. I wish to thank M. Bander, K. Kikkawa, T. Kotani, A. M. Polyakov, and H. Sugawara for many discussions.

## REFERENCES

- <sup>1</sup>V. G. Vaks and A. I. Larkin, Zh. Eksp. Teor. Fiz. 40, 1392 (1961) [Sov. Phys. JETP 13, 979 (1961)]. See also E. Brézin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976), and K. Lane, Phys. Rev. D10, 1353 (1974).
- <sup>2</sup>This model was studied in detail by Vaks and Larkin, Ref. 1 who attribute it to A. A. Anselm, Zh. Eksp. Teor. Fiz. 36, 863 (1959) [Sov. Phys. JETP 9, 608 (1959)]. It has been re-examined in a contemporary light by D. Gross and A. Neveu, Phys. Rev. 10, 3235 (1974) and many of the physics issues associated with spontaneous mass generation have been discussed by Y. Frishman, H. Romer, and S. Yankielowicz, Phys. Rev. D11, 3040 (1975).
- <sup>3</sup>A. M. Polyakov, "Quark Confinement and Toplogy of Gauge Groups," NORDITA Preprint 76/33, October, 1976.

## FIGURE CAPTIONS

- Fig. 1: The one fermion loop approximation to the scalar propagator. This determines  $Z_3$ .
- Fig. 2: The behavior of the function  $F(\lambda)^2$  as a function of  $\lambda$  at  $D = 2$ . The spontaneous fermion mass is  $\mu F(\lambda)$ .
- Fig. 3: The function  $\beta(\lambda)$  for the four fermion coupling model at  $D = 2$ . The infrared stable zero at  $\lambda^2 = \pi$  arises from the self-consistent  $\beta$ -function.

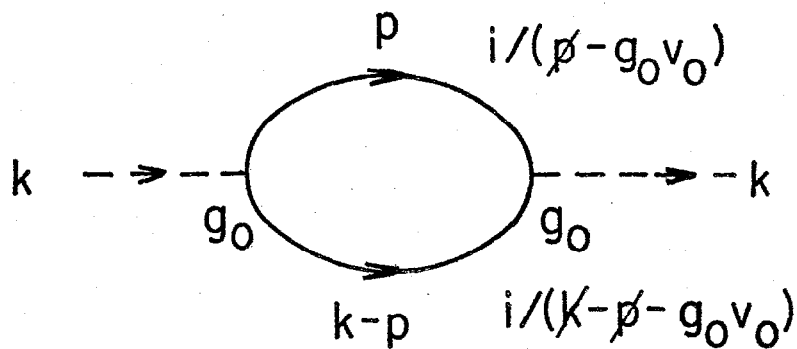


Fig. 1

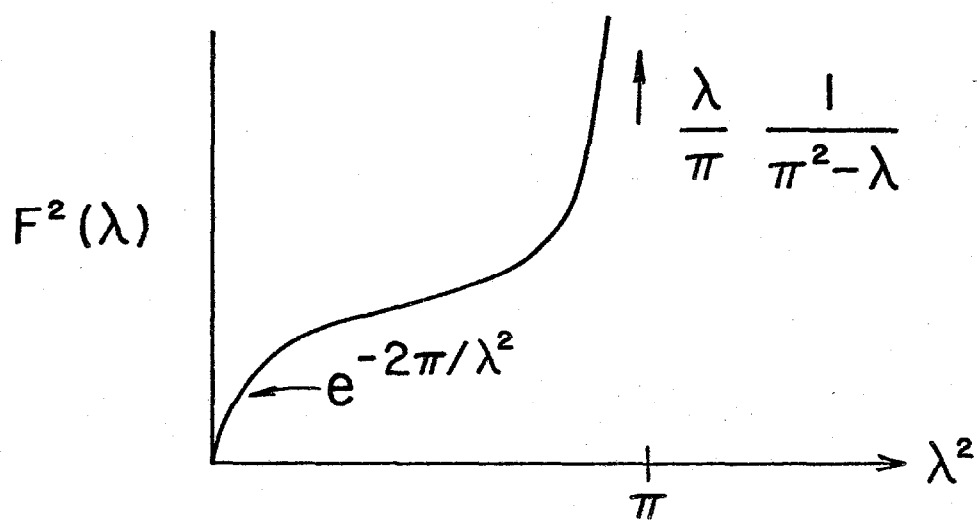


Fig. 2

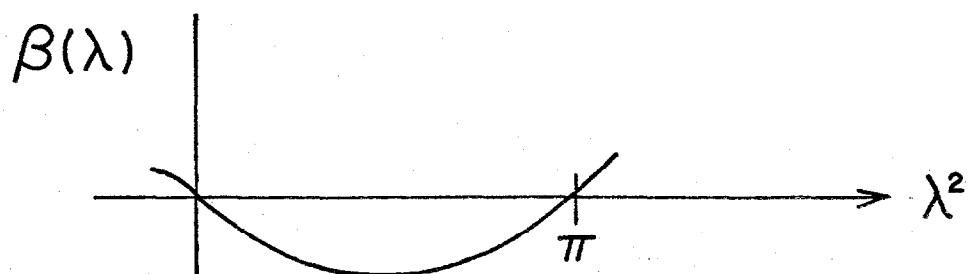


Fig. 3